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The conserved Penrose–Fife system with temperature-dependent memory

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Abstract

A nonlinear parabolic system of Penrose–Fife type with a singular evolution term, arising from modelling dynamic phenomena of the nonisothermal diffusive phase separation, is studied. Here, we consider the evolution of a material in which the heat flux is a superposition of two different contributions: one part is proportional to the spacial gradient of the inverse of the absolute temperature ϑ, while the other agrees with the Gurtin–Pipkin law, introduced in the theory of materials with thermal memory. The phase transition here is described through the evolution of the conserved order parameter χ, which may represent the density or concentration of some substance. It is shown that an initial-boundary value problem for the resulting state equations has a unique solution.

Keywords: Penrose–Fife model; Thermal memory; Gurtin–Pipkin law; Conserved order parameter

1. Introduction

The paper is devoted to the study of certain initial-boundary value problem, which provides a quite general version of the phase-field model proposed by Penrose and Fife in [29] and [30] for the kinetic of phase-transitions. The system of partial differential equations has here the form

$$\partial_t\big(\vartheta + \lambda(\chi)\big) + \operatorname{div} \mathbf{q} = g \quad \text{in } Q, \tag{1.1}$$

$$\partial_t \chi - \Delta\big(-\Delta\chi + \beta(\chi) + \sigma'(\chi) + \lambda'(\chi)/\vartheta\big) \ni 0 \quad \text{in } Q, \tag{1.2}$$

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where Q stands for the cylindrical domain $\Omega \times (0,T)$, Ω is a smooth, bounded, and connected domain in $\mathbb{R}^N$, $1 \leqslant N \leqslant 3$, with smooth boundary $\Gamma := \partial\Omega$, $T > 0$ is the final time of the process.

In (1.1)–(1.2), $\lambda'(\cdot)$ denotes the (in general nonconstant) latent heat density, g is a source term, β is an arbitrary maximal monotone graph in $\mathbb{R}^2$, and $\beta + \sigma'$ stands for the derivative of a possibly nonconvex free energy potential.

Of course, the energy balance equation (1.1) has to be supplied with a constitutive law for the heat flux $\mathbf{q}$ and initial boundary conditions for (1.1)–(1.2) have to be prescribed. In our paper, we assume

$$\mathbf{q} = -\nabla(-\delta/\vartheta + k * \vartheta) \tag{1.3}$$

for some positive constant δ, where $k : [0, +\infty) \to \mathbb{R}$ is a memory kernel, and $*$ stands for the standard convolution product with respect to time

$$(a * b)(t) := \int_0^t a(s)b(t-s)\,ds, \quad t \in [0,T], \tag{1.4}$$

where a and b may also depend on the space variables. Moreover, we keep k as a smooth function, with the only natural restriction that $k(0) > 0$. For a justification of (1.3) we refer to [15] and for other related works, where phase-transitions systems with memory effects are considered, we refer, e.g., to [4,6,10,18] and references therein.

Next, we supply (1.1) with a third type boundary condition

$$\mathbf{q} \cdot \mathbf{n} = \gamma(-\delta/\vartheta + k * \vartheta - h) \quad \text{on } \Sigma := \Gamma \times (0,T), \tag{1.5}$$

where $\mathbf{n}$ indicates the outward normal vector, γ a positive coefficient and the datum h depends on the outside temperature on the boundary.

Moreover, we couple (1.2) with the "natural" homogeneous Neumann boundary condition for both the concentration χ (see [28] for a justification) and the chemical potential $w := -\Delta\chi + \xi + \sigma'(\chi) + \lambda'(\chi)/\vartheta$ (ξ is a selection of $\beta(\chi)$),

$$\partial_{\mathbf{n}}\chi = 0 \quad \text{on } \Sigma \tag{1.6}$$

and

$$\partial_{\mathbf{n}} w = 0 \quad \text{on } \Sigma, \tag{1.7}$$

where $\partial_{\mathbf{n}}$ denotes the outer normal derivative. Finally, the initial conditions

$$\vartheta(\cdot\,,0) = \vartheta_0, \quad \chi(\cdot\,,0) = \chi_0 \quad \text{in } \Omega \tag{1.8}$$

complete the formulation of the problem under study.

Now, we may observe that χ is a "conserved order parameter" because (1.2), together with (1.7), implies that the average of χ (w.r.t. the space variables) does not change in time. Actually, as χ represents the density or concentration of some substance (e.g., one of the components in an alloy), then it follows that the dynamical process can at most move parts of the substance from one place to another, but not create or annihilate masses. In this setting, (1.2) can be seen as a generalization of the Cahn–Hilliard equation (see, e.g., [3,5]). Typical examples of a phase-transition with this kind of dynamics are the phase separation process characteristic of multi-component alloys, polymers, glasses, magnets, and liquid mixtures (cf. [1] and references therein for further details on this subject).

Let us sketch some motivations for our system. First, we have to say that, in the original model (see [29]), the heat flux is assumed to be given by the Fourier law

$$\mathbf{q} = -\varepsilon\nabla\vartheta \tag{1.9}$$

for some $\varepsilon > 0$. This kind of position in the conserved case has recently been studied in [33] for some special cases of nonlinearities in (1.2), that was also taken into account in [25] for the nonconserved case (which basically differs from (1.1) and (1.2) because of a second order dynamics for χ). Always in this framework, but with a nonsmooth choice for the free energy, there is work [14], in which, however, the interfacial energy is set equal to zero (i.e., the Laplacian of χ is neglected in the equation correspondent to our (1.2)).

Several papers have been devoted to the investigation of conserved (or nonconserved) phase field models with

$$\mathbf{q} = -\nabla(-\delta/\vartheta) \tag{1.10}$$

as heat flux law (see, e.g., [19,21,23,34] in the conserved case and [12,16,20,24,35] for the nonconserved one).

However, law (1.10), that turns out to be satisfactory for low and intermediate temperatures and offers some advantages from the mathematical point of view, does not look acceptable for high temperatures because it does not provide any coerciveness as ϑ becomes larger and larger. These considerations suggest to replace (1.10) by (1.3). We have also to notice that the first work in which a nonconserved Penrose–Fife model was coupled with memory is [13], in which just (1.3) is taken into account. Let us observe that another way to overcome such difficulties is to take

$$\mathbf{q} = -\nabla(-\delta/\vartheta + \varepsilon\vartheta) \tag{1.11}$$

as heat flux law, that is satisfactory also for large values of temperatures, and was studied (also in some generalizations), for example, in [9,11,31] for the nonconserved case, and in [32] for the conserved one.

Let us comment the choice to deal with third type boundary conditions for the flux (cf. (1.5) with $\gamma > 0$). This type of boundary conditions are very common in this framework because they are simpler to treat from the mathematical point of view. Instead, only few works deal with the case of Neumann boundary conditions (cf. (1.5) with $\gamma = 0$ or also the corresponding Neumann nonhomogeneous one: $\mathbf{q} \cdot \mathbf{n} = h$). We can cite for the case of Neumann homogeneous boundary conditions on the heat flux $\mathbf{q}$ of the form (1.10) the pioneering paper of Zheng (cf. [36]) and the work [34], which considered respectively the 1D nonconserved and conserved cases with a double-well potential (i.e., with $\beta(\chi) = \chi^3$ in (1.2)). Moreover, the case of nonconserved and conserved models with Neumann nonhomogeneous boundary conditions always for an heat flux of the form (1.10) has been solved in [17] under the null-mean value condition on the given functions g and h, i.e.,

$$\int_\Omega f + \int_\Gamma h = 0$$

during all the evolution. Finally, in paper [8] we have recently proved well posedness and a regularity result for the nonconserved model with Neumann nonhomogeneous boundary conditions for an heat flux which is a generalization of (1.11).

The main aim of this paper is instead to prove existence and uniqueness of a weak solution to (1.1)–(1.3), (1.5)–(1.8), with constant latent heat density λ' and with strictly positive coefficient γ in (1.5). In doing that, we are inspired by [13], because, like in that work, we cannot extract any spacial regularity on ϑ, that might help in treatment of the perturbation due to $k * \vartheta$. Therefore, we first consider the problem where (1.3) is replaced by (1.11) (which we have just solved in [32]). Then, including $\varepsilon\vartheta$ $(\varepsilon > 0)$ in (1.3), we can employ a fixed-point technique to show that such approximating problems admit a unique solution. Finally, we take the limit as $\varepsilon \searrow 0$ to recover a solution (ϑ, χ) of (1.1)–(1.3), (1.5)–(1.8). A uniqueness result is a consequence of a contracting estimate. Let us observe that this is the crucial point of this work, in sense that here the lower time regularity of χ, given by the fourth order equation, has to be supplied by the ones of ϑ and so the conserved problem looks quite different from the analogous nonconserved one (see Remark 3.2 below for more details on this subject). We can observe that results analogous to the ones proved here may be obtained with λ only Lipschitz continuous with its derivatives for the corresponding viscous problem (i.e., with the adjoint of the term $-\Delta(\partial_t \chi)$ in (1.2)). Indeed, in this case we may recover from the energy estimate (i.e., in few words, multiplying (1.1) by $\vartheta - 1/\vartheta$ and summing it up to (1.2) tested with $-\Delta^{-1}(\partial_t \chi)$) more regularity on $\partial_t \chi$, which in this case will turn out to be in $L^2(\Omega)$ and not only in the dual space of $H^1(\Omega)$. This will allow us to consider the duality pairing between $\lambda'(\chi)\partial_t \chi$ and ϑ as a scalar product in $L^2(\Omega)$. Hence, in this case we do not need the function λ' to be constant, but it is sufficient to require only its boundedness. Finally, let us note that the analogous problem of (1.1)–(1.2) with (1.10) as heat flux law may be obtained from this system (i.e., with (1.3) as heat flux law) if we take as memory kernel a Dirac mass in zero, and so it may be interesting to study the behaviour of solutions to (1.1)–(1.2) and (1.3) when k tends to a Dirac mass and it should be the subject of a forthcoming paper.

2. Main result

Consider the initial-boundary value problem (1.1)–(1.2), (1.3)–(1.8). We make the following general assumptions on the data of the system:

$$\begin{aligned}&\beta \text{ is the subdifferential of a nonnegative, proper, convex,}\\ &\quad \text{and l.s.c. function } \hat\beta : \mathbb{R} \to [0, +\infty] \text{ satisfying } \hat\beta(0) = 0,\\ &\quad \text{denote by } K \text{ the closure of } D(\beta) \text{ with } D(\beta) \text{ the domain of } \beta \text{ in } \mathbb{R},\end{aligned} \tag{2.1}$$

$$\sigma \in C^1(K), \quad \sigma' \in C^{0,1}(K), \quad \text{be } C_\sigma := \|\sigma''\|_{L^\infty(K)}, \tag{2.2}$$

$$\lambda : \mathbb{R} \to \mathbb{R} \quad \text{and} \quad \lambda'(r) = \ell, \quad \forall r \in K \text{ and for some } \ell \in \mathbb{R}, \tag{2.3}$$

$$k \in W^{2,1}(0, T) \quad \text{with } k(0) > 0, \tag{2.4}$$

$$g \in L^2(Q), \quad h \in L^2(\Sigma) \quad \text{with } h \leqslant 0 \text{ a.e. in } \Sigma, \tag{2.5}$$

$$\vartheta_0 \in L^2(\Omega), \quad \vartheta_0 > 0 \quad \text{a.e. in } \Omega, \quad \log(\vartheta_0) \in L^1(\Omega), \tag{2.6}$$

$$\chi_0 \in H^1(\Omega), \quad \hat{\beta}(\chi_0) \in L^1(\Omega). \tag{2.7}$$

Now let us give a variational formulation of (1.1)–(1.3), (1.5)–(1.8). To this end, we denote by $(\cdot\,,\cdot)$ both the scalar product in $H := L^2(\Omega)$ and in $(L^2(\Omega))^N$, also denoted by H, and by $|\cdot|$ the corresponding norm. For the sake of convenience, $V := H^1(\Omega)$ will be endowed with the inner product $((\cdot\,,\cdot))$, defined by

$$((v_1, v_2)) := \int_\Omega \nabla v_1 \nabla v_2 + \gamma \int_\Gamma v_1 v_2, \quad \forall v_1, v_2 \in V, \tag{2.8}$$

where γ is the positive constant appearing in the boundary condition (1.5). Define $W := H^2(\Omega)$ and let us also indicate by $\langle\cdot\,,\cdot\rangle$ the duality pairing between V' and V. We identify H with a subspace of V', as usual, so that $\langle u, v\rangle = (u, v)$ for all $u \in H$ and for all $v \in V$.

Next, we define the Riesz isomorphism $J : V \to V'$, and the scalar product in V', respectively, by

$$\langle J v_1, v_2\rangle := ((v_1, v_2)), \quad \forall v_1, v_2 \in V, \tag{2.9}$$

$$((w_1, w_2))_* := \langle w_1, J^{-1} w_2\rangle, \quad \forall w_1, w_2 \in V'. \tag{2.10}$$

Note that, if $v \in V$, $Jv \in H$, and $w \in H$, then it holds that

$$(Jv, J^{-1}w) = ((Jv, w))_* = ((w, Jv))_* = \big(w, J^{-1}(Jv)\big) = (w, v). \tag{2.11}$$

Let us observe that the norm in V related to the inner product defined above (which will be indicated as $\|\cdot\|$) is equivalent to the usual norm in V. Similar considerations hold also for V' and we term $\|\cdot\|_*$ the norm in V' related to the inner product (2.10).

We may now introduce the following notations:

$$\psi_\Omega := \frac{1}{|\Omega|}\langle\psi, 1\rangle, \quad \forall \psi \in V', \tag{2.12}$$

$$\varphi_\Gamma := \frac{1}{|\Gamma|}\int_\Gamma \varphi, \quad \forall \varphi \in V, \tag{2.13}$$

and the following spaces:

$$\mathcal{V} = \{v \in V,\ \text{such that } v_\Omega = 0\}, \qquad \mathcal{H} = \{v \in H,\ \text{such that } v_\Omega = 0\},$$
$$\mathcal{W} = \{v \in W,\ \text{such that } \partial_{\mathbf{n}} v = 0 \text{ on } \Gamma \text{ and } v_\Omega = 0\},$$
$$\mathcal{V}' := \{v \in V',\ \text{such that } v_\Omega = 0\}.$$

We may define now the operator $\mathcal{N} : \mathcal{H} \to \mathcal{W}$ that maps $v \in \mathcal{H}$ into the unique function $\mathcal{N}v \in \mathcal{W}$ such that

$$-\Delta(\mathcal{N}v) = v \quad \text{a.e. in } \Omega, \qquad \partial_{\mathbf{n}}(\mathcal{N}v) = 0 \quad \text{a.e. on } \Gamma, \quad \text{and} \quad \int_\Omega \mathcal{N}v = 0.$$

Note that any solution ϕ to

$$-\Delta\phi = v \quad \text{a.e. in } \Omega \quad \text{and} \quad \partial_{\mathbf{n}}\phi = 0 \quad \text{a.e. on } \Gamma, \tag{2.14}$$

corresponding to $v \in \mathcal{H}$, can be written as $\phi = \mathcal{N}v + \mu$, where μ is the mean-value of ϕ.

The operator $\mathcal{N}$ is an isomorphism and it may be extended to a new operator (always called $\mathcal{N}$) from $\mathcal{V}'$ to $\mathcal{V}$ (note the space $\mathcal{V}'$ may not be identified with the dual space of $\mathcal{V}$), such that

$$\mathcal{N}v \in \mathcal{V}, \quad \int_\Omega \nabla\big(\mathcal{N}v\big)\nabla z = \langle v, z\rangle, \quad \forall z \in \mathcal{V}. \tag{2.15}$$

Note that $\mathcal{N}$ is also an isomorphism from $\mathcal{V}'$ to $\mathcal{V}$, so that, for $v \in \mathcal{V}'$, the norm

$$\|v\|_\star := \left(\int_\Omega \big|\nabla(\mathcal{N}v)\big|^2\right)^{1/2} = \langle v, \mathcal{N}v\rangle^{1/2} \tag{2.16}$$

is equivalent to the norm $\|v\|_*$, i.e., there exist two positive constants $C_* \leqslant C_\star$ (depending only on Ω) s.t.

$$C_*\|v\|_*^2 \leqslant \|v\|_\star^2 \leqslant C_\star\|v\|_*^2, \quad \forall v \in \mathcal{V}'. \tag{2.17}$$

Finally, let $f \in L^2(0,T;V')$ be defined by

$$\big\langle f(t), v\big\rangle := \int_\Omega g(t)v + \gamma\int_\Gamma h(t)v, \quad \forall v \in V \text{ and for a.e. } t \in (0,T). \tag{2.18}$$

We are now ready to state the rigorous formulation of our problem, in which we can assume the constant δ of (1.3) and (1.5) equal to 1 for simplicity of notations and without loss of generality.

Problem (P_0). Find a pair (ϑ,χ) and (w,ξ) such that

$$\vartheta \in H^1(0,T;V') \cap L^\infty(0,T;H), \quad \vartheta > 0 \text{ a.e. in } Q, \tag{2.19}$$

$$u := -\frac{1}{\vartheta} \in L^2(0,T;V), \quad k * \vartheta \in L^\infty(0,T;V), \tag{2.20}$$

$$\chi \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap C^0\big([0,T];H\big), \tag{2.21}$$

$$\xi \in L^2(0,T;H), \qquad w \in L^2(0,T;V), \tag{2.22}$$

$$\xi \in \beta(\chi) \quad \text{a.e. in } Q, \qquad \chi \in D(\beta) \quad \text{a.e. in } Q, \tag{2.23}$$

$$\partial_t\vartheta + \ell\partial_t\chi + Ju + J(k*\vartheta) = f \quad \text{in } V', \text{ a.e. in } (0,T), \tag{2.24}$$

$$\langle\partial_t\chi, v\rangle + \int_\Omega \nabla w\nabla v = 0, \quad \forall v \in V, \text{ a.e. in } (0,T), \tag{2.25}$$

$$\langle w, v\rangle = \int_\Omega \nabla\chi\nabla v + \big\langle \xi + \sigma'(\chi) - \ell u, v\big\rangle, \quad \forall v \in V, \text{ a.e. in } (0,T), \tag{2.26}$$

$$\vartheta(\cdot,0) = \vartheta_0, \quad \chi(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{2.27}$$

Remark 2.1. The first relation in (2.20) may also be rewritten as $\vartheta \in \rho(u)$, where ρ is the maximal monotone graph defined in $(-\infty,0)$ by $\rho(r) = -1/r$ for $r > 0$. Note that this representation will be useful especially in the proof (cf. Section 3) of the last Theorem 2.6 of this section.

Remark 2.2. Let us note that testing (2.25) with $v = 1$ yields

$$\frac{d}{dt}\langle \chi, 1\rangle = 0 \quad \text{in } (0, T).$$

This means that (recall notation (2.12))

$$\chi_\Omega = (\chi_0)_\Omega =: m_0, \quad \forall t \in [0, T], \tag{2.28}$$

i.e., the mean value of χ is conserved. This fact is often used in the sequel.

Our main result is the following

Theorem 2.3. *Suppose that* (2.1)–(2.7) *hold. Moreover suppose that*

$$m_0 \in \operatorname{int} K \tag{2.29}$$

with m_0 defined as in (2.28)*. Then* (P_0) *has at least a solution and the components ϑ, χ of such a solution are unique.*

In order to get a proof of this theorem, we introduce a family of approximating Problem (P_ε), which contain (P_0) as special case for $\varepsilon = 0$, and then we pass to the limit as $\varepsilon \searrow 0$.

Problem (P_ε). For fixed $\varepsilon \geqslant 0$, find a pair (ϑ, χ) and (w, ξ) satisfying the conditions of (P_0), where (2.24) is replaced by

$$\partial_t \vartheta + \ell \partial_t \chi + J(u + \varepsilon\vartheta) + J(k * \vartheta) = f \quad \text{in } V', \text{ a.e. in } (0, T), \tag{2.30}$$

$$\varepsilon\vartheta \in L^2(0, T; V). \tag{2.31}$$

Then the existence–uniqueness theorem for the approximating problems is

Theorem 2.4. *Suppose that* (2.1)–(2.7) *and* (2.29) *are satisfied. Then, for any $\varepsilon > 0$ and sufficiently small,* (P_ε) *has at least a solution and the components ϑ, χ of such a solution are unique.*

Remark 2.5. In case of $\varepsilon > 0$, it suffice that $k \in L^2(0, T)$ and $f \in L^2(0, T; V')$, and so hypothesis (2.4)–(2.5) may be omitted, as pointed out in the Remark 4.2 and Lemmas 4.3–4.5 below.

In order to give a proof of Theorem 2.4, we need to introduce a further family of problems, corresponding to the case $k = 0$, that have to be studied separately.

Problem $(\mathrm{P}'_\varepsilon)$. Let $\varepsilon \geqslant 0$ and F only belong to $L^2(0, T; V')$. Find a pair (ϑ, χ) and (w, ξ) satisfying the conditions of (P_ε), where (2.30) is replaced by

$$\partial_t \vartheta + \ell \partial_t \chi + J(u + \varepsilon\vartheta) = F \quad \text{in } V', \text{ a.e. in } (0, T). \tag{2.32}$$

Here we have the following existence–uniqueness result related to this problem.

Theorem 2.6. *Assume hypothesis* (2.1)–(2.3), (2.6)–(2.7), (2.29) *and suppose moreover that* $F \in L^2(0,T;V')$. *Then* $(\mathrm{P}'_\varepsilon)$ *admits at least a solution for any* $\varepsilon > 0$ *and sufficiently small. Moreover, the components* ϑ, χ *of this solution are unique. If in addition* $F = f$, *with* f *defined in* (2.18) *and such that* (2.5) *holds, then also* (P'_0) *has at least a solution and the components* ϑ, χ *of such a solution are unique.*

Remark 2.7. The proof of the first part of Theorem 2.6, i.e., the existence–uniqueness of a solution to Problem $(\mathrm{P}'_\varepsilon)$ for $\varepsilon > 0$, can be found in [32]. More in detail, using [32, Theorems 2.1 and 2.2] (with the kernel of convolution identically equal to 0), we may get existence of a solution to Problem $(\mathrm{P}'_\varepsilon)$ (for $\varepsilon > 0$), and uniqueness of the components ϑ and χ of such a solution. We have also to observe that the uniqueness result in [32] is achieved only under strong regularity assumptions on the data, but in [32] the equation correspondent to our (2.32) contained also a memory term. In this paper instead, we will prove a continuous dependence result on solution of $(\mathrm{P}'_\varepsilon)$ (for $\varepsilon > 0$ sufficiently small) and so uniqueness for the component ϑ and χ, with datum $F \in L^2(0,T;V')$ and with regularity assumptions (2.6) and (2.7) on the data (see Lemma 3.1 below).

Remark 2.8. In the case $\varepsilon = 0$, Theorem 2.6 is a generalization of [23], in which the authors find existence of a solution (only in case of β with bounded domain) and uniqueness (without any adjoint regularity assumption on solutions) only in dimension $N = 1$. Moreover in [19] uniqueness of solution also in dimension N (with $1 \leqslant N \leqslant 3$) is recovered, always in case of β with bounded domain and convex λ. In [34] existence and uniqueness of solution to (P'_0) are given for some special cases of nonlinearities in (2.26) and only in dimension $N = 1$.

To conclude this section, let us recall these two formulas concerning the convolution product which hold whenever they make sense, namely the identities

$$a * b = a(0)(1 * b) + a' * (1 * b), \tag{2.33}$$

$$(a * b)' = a(0)b + a' * b, \tag{2.34}$$

and the Young theorem

$$\|a * b\|_{L^r(0,T;X)} \leqslant \|a\|_{L^p(0,T)} \|b\|_{L^q(0,T;X)} \tag{2.35}$$

with $1 \leqslant p, q, r \leqslant \infty$, $1/r = 1/p + 1/q - 1$, where X is a normed space.

Moreover, we remember that (see, e.g., [27, Theorem 16.4, p. 102])

$$|v|^2 + \|v\|^2_{L^4(\Omega)} \leqslant \zeta \|v\|^2 + C_\zeta \|v\|^2_*, \quad \forall v \in V, \tag{2.36}$$

for any $\zeta > 0$ and some constant $C_\zeta > 0$.

Let us remember at this point that, by the continuity of the trace operator (in this setting) from V to $L^2(\Gamma)$, there exists a positive constant C_Γ (depending only on Ω) such that

$$\|v\|^2_{L^2(\Gamma)} \leqslant C_\Gamma \|v\|^2. \tag{2.37}$$

We widely use also the elementary inequality

$$ab \leqslant \mu a^2 + \frac{1}{4\mu} b^2, \quad \forall a, b \in \mathbb{R},\ \forall \mu > 0. \tag{2.38}$$

Finally, we have to recall a property of the function "ln," which will be useful in the sequel: if $a, b > 0$ are given, then there exists a positive constant C such that

$$ar^2 - 2b\ln(r) + C \geqslant \frac{a}{2}r^2 + 2b\left|\ln(r)\right|, \quad \forall r > 0. \tag{2.39}$$

3. Analysis of Problem ($\mathbf{P}'_\varepsilon$)

The aim of this section is to prove Theorem 2.6. It is just known by [32] that (P'_ε) for $\varepsilon > 0$ has at least a solution (see also Remark 2.7) in hypothesis (2.1)–(2.7) on the data. We want to get uniqueness of solution for (P'_ε) ($\varepsilon \geqslant 0$) and then existence of solution to Problem (P'_0). Let us first consider Problem (P'_ε) for $\varepsilon \geqslant 0$ and begin with a "continuous dependence" result

Lemma 3.1. *Let $0 \leqslant \varepsilon < 1$ and suppose that (ϑ_i, χ_i), (w_i, ξ_i) denote solutions to* (P'_ε) *corresponding to the data $F_i, \vartheta_{0i}, \chi_{0i}$, $i = 1, 2$. Let*

$$\eta_i = \vartheta_i + \lambda(\chi_i), \quad \eta_{0i} = \vartheta_{0i} + \ell\chi_{0i}, \quad m_{0i} = (\chi_{0i})_\Omega, \quad i = 1, 2, \tag{3.1}$$

$$\chi = \chi_1 - \chi_2, \quad \eta = \eta_1 - \eta_2, \quad \vartheta = \vartheta_1 - \vartheta_2, \quad u = u_1 - u_2, \tag{3.2}$$

$$\eta_0 = \eta_{01} - \eta_{02}, \quad \chi_0 = \chi_{01} - \chi_{02}, \quad F = F_1 - F_2, \tag{3.3}$$

and suppose that

$$m_{01} = m_{02}. \tag{3.4}$$

Then there exists some positive constant C (depending only on the data) such that

$$\begin{aligned}&\left\|\eta(t)\right\|_*^2 + \varepsilon\|\vartheta\|^2_{L^2(0,t;H)} + C_*\left\|\chi(t)\right\|_*^2 + \|\chi\|^2_{L^2(0,t;V)}\\ &\quad\leqslant \|\eta_0\|_*^2 + C_\star\|\chi_0\|_*^2 + C\int_0^t \|\chi\|_*^2 + 2\int_0^t \langle F, J^{-1}\eta\rangle, \quad \forall t \in [0, T].\end{aligned} \tag{3.5}$$

In particular, the components ϑ and χ of the solutions of (P'_ε) *are unique.*

Proof. Subtract the respective equations (2.32) for (ϑ_i, χ_i) (with $i = 1, 2$) from each other, multiply the result by $J^{-1}\eta$, and integrate over $(0, t)$. Then, thanks to (3.4), we can choose $v = \mathcal{N}\chi_i$ in (2.25) (for $i = 1, 2$), sum it to (2.26) (for $i = 1, 2$) tested with $v = -\chi_i$, take the difference, and integrate on $(0, t)$. In such a way, we can find that, using the monotonicity of β, (2.10), and (2.16),

$$\begin{aligned}&\frac{1}{2}\left\|\eta(t)\right\|_*^2 + \int_0^t \langle Ju, J^{-1}\eta\rangle + \varepsilon\int_0^t \langle J\vartheta, J^{-1}\eta\rangle + \frac{1}{2}\left\|\chi(t)\right\|_\star^2\\ &\quad + \int_0^t\int_\Omega |\nabla\chi|^2 - \ell\int_0^t (u, \chi)\end{aligned}$$

$$\leqslant \frac{1}{2}\|\eta_0\|_*^2 + \frac{1}{2}\|\chi_0\|_\star^2 - \int_0^t \int_\Omega \big(\sigma'(\chi_1) - \sigma'(\chi_2)\big)\chi + \int_0^t \langle F, J^{-1}\eta\rangle. \tag{3.6}$$

Now, we may estimate separately the single parts of the previous inequality. Hence, using the monotonicity of the function $\vartheta \mapsto -1/\vartheta$ we may conclude that

$$\int_0^t \langle Ju, J^{-1}\eta\rangle - \ell \int_0^t \int_\Omega u\chi = \int_0^t (u, \vartheta) \geqslant 0. \tag{3.7}$$

Moreover, thanks to (2.11), Hölder inequality, and (2.36), we have that for all $t \in [0, T]$,

$$\begin{aligned}
\varepsilon \int_0^t \langle J\vartheta, J^{-1}\eta\rangle &= \varepsilon\|\vartheta\|^2_{L^2(0,t;H)} + \varepsilon\ell \int_0^t (\vartheta, \chi) \\
&\geqslant \left(\varepsilon - \frac{\varepsilon^2}{2}\right)\|\vartheta\|^2_{L^2(0,t;H)} - \frac{1}{2}\ell^2\|\chi\|^2_{L^2(0,t;H)} \\
&\geqslant \left(\varepsilon - \frac{\varepsilon^2}{2}\right)\|\vartheta\|^2_{L^2(0,t;H)} - \ell^2\delta\|\chi\|^2_{L^2(0,t;V)} - \ell^2 C_\delta\|\chi\|^2_{L^2(0,t;V')}
\end{aligned} \tag{3.8}$$

for all $\delta > 0$ and for some constant $C_\delta > 0$.

Next, (2.2) ensures that, always using (2.36) after Schwartz inequality, the following inequality holds for all $t \in [0, T]$:

$$-\int_0^t \int_\Omega \big(\sigma'(\chi_1) - \sigma'(\chi_2)\big)\chi \leqslant C_\sigma\delta'\|\chi\|^2_{L^2(0,t;V)} + C_\sigma C_{\delta'}\|\chi\|^2_{L^2(0,t;V')}, \quad \forall\delta' > 0, \tag{3.9}$$

and for some positive constant $C_{\delta'}$.

Note that, by Poincaré inequality and thanks to (3.4), $\|\nabla\chi\|_{L^2(0,t;H)}$ is equivalent to $\|\chi\|_{L^2(0,t;V)}$ and so, combining (3.7)–(3.9), choosing δ and δ' sufficiently small and $\varepsilon < 1$, and using (2.17) in (3.6), we obtain (3.5), whence the uniqueness result easily follows using Gronwall's lemma. Observe that this conclusion holds also in case of $\varepsilon = 0$. □

In the remaining of this section we derive some estimates for $(\mathrm{P}'_\varepsilon)$ uniform in ε in order to pass to the limit in $(\mathrm{P}'_\varepsilon)$ as $\varepsilon \searrow 0$ and so prove the last part of Theorem 2.6. So, let $F = f$ be as in (2.18). We shall denote by C any positive constant (possibly not the same even inside the same row) that only depends on Ω, N, T, ϑ_0, χ_0, and in particular not on ε and $t \in [0, T]$. We will denote by symbols υ_t the time derivative of the generic variable υ.

First estimate. Let now $(\vartheta, \chi, w, \xi)$ be a solution of $(\mathrm{P}'_\varepsilon)$ for $\varepsilon > 0$, since u and ϑ belongs to $L^2(0, T; V)$ and recalling notation (2.12), we may multiply both sides of (2.32) by $u + \delta\vartheta + \delta\mathcal{N}(\vartheta_t - (\vartheta_t)_\Omega)$ for some positive constant δ that we will choose later.

Moreover, we test Eqs. (2.25) and (2.26) with $\mathcal{N}\chi_t$ and $-\chi_t$, respectively (see Remark 2.2), and then take the sum of the three resulting equations, use (2.17), and integrate over $(0, t)$, obtaining

$$
\begin{aligned}
&\int_\Omega \left(\frac{\delta\vartheta^2}{2} - \ln(\vartheta)\right)(t) + C_*\delta \left\|\vartheta_t - (\vartheta_t)_\Omega\right\|^2_{L^2(0,t;V')} + \|u\|^2_{L^2(0,t;V)} + \varepsilon\delta\|\vartheta\|^2_{L^2(0,t;V)} \\
&\quad + (\delta+\varepsilon)\int_0^t\int_\Omega \frac{|\nabla\vartheta|^2}{\vartheta^2} + \|\chi_t\|^2_{L^2(0,t;V')} + \frac{1}{2}\left|\nabla\chi(t)\right|^2 + \int_\Omega \hat\beta(\chi)(t) \\
&\leqslant \int_\Omega \left(\frac{\delta\vartheta_0^2}{2} - \ln(\vartheta_0)\right) + \gamma\int_0^t\int_\Gamma (\delta+\varepsilon) - \ell\int_0^t \langle\chi_t, u+\delta\vartheta\rangle \\
&\quad - \ell\delta\int_0^t \left\langle \chi_t, \mathcal{N}\big(\vartheta_t - (\vartheta_t)_\Omega\big)\right\rangle \\
&\quad - \delta\int_0^t \Big(\big(u+\varepsilon\vartheta, \mathcal{N}\big(\vartheta_t - (\vartheta_t)_\Omega\big)\big)\Big) + \frac{1}{2}|\nabla\chi_0|^2 + \int_\Omega \hat\beta(\chi_0) - \int_0^t \left\langle\chi_t, \sigma'(\chi)\right\rangle \\
&\quad + \ell\int_0^t\langle\chi_t, u\rangle + \delta\int_0^t\langle f, \vartheta\rangle + \int_0^t \left\langle f, u + \delta\mathcal{N}\big(\vartheta_t - (\vartheta_t)_\Omega\big)\right\rangle, \quad \forall t\in[0,T],
\end{aligned} \tag{3.10}
$$

thanks also to (2.16), (2.1), and (2.8).

Now, we may estimate every term of (3.10), separately. First, we use (2.39) with $a = \delta/2$ and $b = 1/2$ and get

$$
\int_\Omega \left(\frac{\delta\vartheta^2}{2} - \ln(\vartheta)\right)(t) \geqslant \frac{\delta}{4}\left|\vartheta(t)\right|^2 + \left\|\ln(\vartheta)(t)\right\|_{L^1(\Omega)} - C. \tag{3.11}
$$

Thanks to (2.28) and the definition of J, testing (2.30) with $v = 1$ and recalling notations (2.12) and (2.13), we may find that

$$
|\Omega|(\vartheta_t)_\Omega(s) = |\Omega| f_\Omega(s) - \gamma|\Gamma|\big(u_\Gamma(s) + \varepsilon\vartheta_\Gamma(s)\big)
$$

for all $s \in [0,T]$, and the following estimate holds:

$$
\left\|(\vartheta_t)_\Omega\right\|_* \leqslant C\big(\|f\|_* + C_\Gamma\|u\| + \varepsilon C_\Gamma\|\vartheta\|\big), \tag{3.12}
$$

thanks to (2.37).

Moreover, integrating by parts, using (3.12) with (2.38), we may obtain

$$
\begin{aligned}
-\ell\delta\int_0^t\langle\chi_t,\vartheta\rangle &= \ell\delta\int_0^t\left\langle\vartheta_t - (\vartheta_t)_\Omega, \chi\right\rangle - \ell\delta\int_\Omega \chi\vartheta + \ell\delta\int_\Omega \chi_0\vartheta_0 \\
&\quad + \ell\delta\int_0^t \left\langle(\vartheta_t)_\Omega, \chi\right\rangle \leqslant \ell\delta\gamma'\left\|\vartheta_t - (\vartheta_t)_\Omega\right\|^2_{L^2(0,t;V')} \\
&\quad + \ell\delta C_{\gamma'}\|\chi\|^2_{L^2(0,t;V)} + \ell\delta\gamma''\left|\vartheta(t)\right|^2 + \ell\delta C_{\gamma''}\left|\chi(t)\right|^2
\end{aligned}
$$

$$+\|f\|^2_{L^2(0,t;V')}+\frac{1}{4}\|u\|^2_{L^2(0,t;V)}+\frac{\varepsilon^2\delta}{4}\|\vartheta\|^2_{L^2(0,t;V)}$$
$$+C\delta\|\chi\|^2_{L^2(0,t;V)},\quad \forall\gamma',\gamma''>0, \tag{3.13}$$

for all $t\in[0,T]$ and for some positive constants $C_{\gamma'}$ and $C_{\gamma''}$. Still using (2.15), (2.17), and (2.38), we can immediately obtain for all $t\in[0,T]$,

$$-\ell\delta\int_0^t \langle\chi_t,\mathcal{N}(\vartheta_t-(\vartheta_t)_\Omega)\rangle$$
$$\leqslant \ell\delta\zeta\,\|\vartheta_t-(\vartheta_t)_\Omega\|^2_{L^2(0,t;V')}+\ell\delta C_\zeta\|\chi_t\|^2_{L^2(0,t;V')} \tag{3.14}$$

for all $\zeta>0$ and for some positive constant C_ζ. Moreover, using again (2.17) and (2.38) with Schwartz inequality, we get

$$-\delta\int_0^t \big(\big(u,\mathcal{N}(\vartheta_t-(\vartheta_t)_\Omega)\big)\big)$$
$$\leqslant \frac{1}{4}\|u\|^2_{L^2(0,t;V)}+C_\star\delta^2\|\vartheta_t-(\vartheta_t)_\Omega\|^2_{L^2(0,t;V')}, \tag{3.15}$$
$$-\delta\int_0^t \big(\big(\varepsilon\vartheta,\mathcal{N}(\vartheta_t-(\vartheta_t)_\Omega)\big)\big)$$
$$\leqslant \frac{\varepsilon^2\delta}{4}\|\vartheta\|^2_{L^2(0,t;V)}+C_\star\delta\|\vartheta_t-(\vartheta_t)_\Omega\|^2_{L^2(0,t;V')}, \tag{3.16}$$
$$\int_0^t \langle f,u+\delta\mathcal{N}(\vartheta_t-(\vartheta_t)_\Omega)\rangle$$
$$\leqslant C_\star\delta\zeta'\|\vartheta_t-(\vartheta_t)_\Omega\|^2_{L^2(0,t;V')}+\delta C_{\zeta'}\|f\|^2_{L^2(0,t;V')},\quad\forall\zeta'>0, \tag{3.17}$$

for all $t\in[0,T]$ and for some positive constant $C_{\zeta'}$. Now, thanks to (2.2), we obtain

$$-\int_0^t\langle\chi_t,\sigma'(\chi)\rangle\leqslant C_\sigma+\frac{C_\sigma}{2}\|\chi\|^2_{L^2(0,t;V)}+\frac{1}{2}\|\chi_t\|^2_{L^2(0,t;V')}. \tag{3.18}$$

Finally, here it is essential to recall the form of f, that is

$$\int_0^t\langle f,\vartheta\rangle\leqslant\int_0^t|g||\vartheta|+\gamma\int_0^t\int_\Gamma h\vartheta, \tag{3.19}$$

and we note that, by virtue of (2.5), the latter summand is nonpositive. And so, we may collect (3.10)–(3.19), add to both members of (3.10),

$$\frac{1}{2}|\chi(t)|^2=\frac{1}{2}|\chi_0|^2+\int_0^t\langle\chi_t,\chi\rangle$$

in order to recover the standard V-full norm of χ on the left side (observe that it is equivalent to our $\|\cdot\|$), choose $\varepsilon < 1, \gamma', \gamma'', \zeta, \zeta'$ and then δ sufficiently small, apply a generalized version of Gronwall's lemma, getting finally the estimate

$$\begin{aligned}&\left|\vartheta(t)\right|^2 + \left\|\ln(\vartheta)(t)\right\|_{L^1(\Omega)} + \|\vartheta_t\|^2_{L^2(0,t;V')} + \|u\|^2_{L^2(0,t;V)} + \varepsilon\|\vartheta\|^2_{L^2(0,t;V)}\\ &\quad + \|\chi_t\|^2_{L^2(0,t;V')} + \left\|\chi(t)\right\|^2\\ &\leqslant C\left(1+\varepsilon+\|f\|^2_{L^2(0,t;V')}\right), \quad \forall t\in[0,T]. \end{aligned} \tag{3.20}$$

Remark 3.2. Observe that the first estimate is crucial for this kind of problems (i.e., in energy balance equation without any kind of coerciveness in the heat flux law, as in (2.32) with $\varepsilon = 0$). It is essential in fact in this cases to test the first equation by $\vartheta + u$ (not only by u) and moreover to recover time regularity on ϑ (in order to give an estimate of terms which couples ϑ and $\partial_t\chi$, as we have done above), because it is impossible to get some regularity on $\partial_t\chi$ by the fourth order equation which rules the phase evolution. Obviously this is not the case of the nonconserved case in which it is possible to recover more regularity on $\partial_t\chi$ by the second order equation which rules the evolution of the phase variable in that kind of problems.

Second estimate. Now, in order to prove Theorem 2.6, we need to estimate the $L^2(0,T;H)$ norm of $\beta(\chi)$, with χ a component of the solution to Problem $(\mathrm{P}'_\varepsilon)$ $(\varepsilon > 0)$, independently of ε. So, if (ϑ,χ), (w,ξ) is a solution to Problem $(\mathrm{P}'_\varepsilon)$, with $\varepsilon > 0$, test (2.25) with $\mathcal{N}(\xi-\xi_\Omega)$, and (2.26) with $(\xi-\xi_\Omega)$ (remember notation (2.12)). Here we have to notice that, in order to perform this estimate, we need some regularity on $\xi-\xi_\Omega$, and so, to make this calculation formal, we would have taken an approximation of β in (2.26), for example its Yosida approximation β_τ (for $\tau > 0$). In this way in fact we would have that $\xi_\tau = \beta_\tau(\chi) \in L^\infty(0,T;V)$, thanks to (3.20) and the Lipschitz continuity of β_τ. Since the proof that this approximating problem (for $\tau > 0$) admits a solution and then the passage to the limit as $\tau \searrow 0$ has just been done in [32], we will argue here directly on (2.25)–(2.26) and (2.32). Then, using (2.28), subtracting the resulting equations, setting

$$G = -\mathcal{N}(\chi_t) - \sigma'(\chi) - \ell u,$$

and subtracting also $\langle \xi_\Omega, \xi-\xi_\Omega\rangle = 0$, we obtain the identity

$$\left(\nabla\chi, \nabla(\xi-\xi_\Omega)\right) + |\xi-\xi_\Omega|^2 = (G-\xi_\Omega, \xi-\xi_\Omega) = (G, \xi-\xi_\Omega).$$

Since the first term on the left-hand side is nonnegative, due to the monotonicity of β we deduce that

$$|\xi-\xi_\Omega| \leqslant |G|.$$

Then, thanks to the first estimates, we immediately get

$$\|\xi-\xi_\Omega\|_{L^2(0,T;H)} \leqslant C.$$

In the next step, we would like to derive an analogous estimate for $\beta(\chi)$. To do that, we have to find an upper bound for the L^2-norm of ξ_Ω.

Repeating exactly the argument reported, for example, in [7, Section 4], which follows closely the proof devised by Kenmochi et al. in [22, Lemma 5.2], we can state that

$$\|\xi\|_{L^2(0,T;H)} \leqslant C. \tag{3.21}$$

Note that assumption (2.29) is used at this step.

Now, in order to derive an estimate of w in $L^2(0,T;V)$, we may observe that, thanks to (2.25), $w - w_\Omega$ is a solution of a problem like (2.14) with datum $\partial_t\chi \in L^2(0,T;V')$ (thanks to (3.20)). Hence, estimating the mean value of w with the help of (2.26) (choose $v=1$) and using again (3.20), we can say that

$$\|w\|_{L^2(0,T;V)} \leqslant C. \tag{3.22}$$

Moreover, applying the same argument to (2.26), we obtain that $\chi - m_0$ is the solution of a problem like (2.14) with datum in $L^2(0,T;H)$ and consequently we may obtain that

$$\|\chi\|_{L^2(0,T;W)} \leqslant C. \tag{3.23}$$

Passage to the limit. We aim to obtain uniqueness on the components ϑ and χ of the solution (ϑ,χ), (w,ξ) to (P_0') by passage to the limit in $(\mathrm{P}_\varepsilon')$ as $\varepsilon \searrow 0$. We have just obtained (in the first–second estimates) uniform bounds (i.e., independent on ε) on solution $(\vartheta_\varepsilon,\chi_\varepsilon)$, $(w_\varepsilon,\xi_\varepsilon)$ of Problem $(\mathrm{P}_\varepsilon')$ for $0<\varepsilon<1$.

Now, the standard weak compactness results, the well-known Ascoli theorem, and Aubin's lemma (cf., e.g., [26, p. 58]) permit us to find a pair (ϑ,χ), (w,ϕ) as the weak, or weak-$*$, limit of a suitable subsequence of $(\vartheta_\varepsilon,\chi_\varepsilon)$, $(w_\varepsilon,\xi_\varepsilon)$ (indeed, all the convergence relations below will be intended to hold up to the extraction of subsequences), i.e., we may say that

$$\vartheta_\varepsilon \to \vartheta \quad \text{weakly star in } H^1(0,T;V') \cap L^\infty(0,T;H) \tag{3.24}$$

$$\text{and strongly in } C^0\big([0,T];V'\big), \tag{3.25}$$

$$\varepsilon\vartheta_\varepsilon \to 0 \quad \text{strongly in } L^2(0,T;V), \tag{3.26}$$

$$u_\varepsilon \to u \quad \text{weakly in } L^2(0,T;V), \tag{3.27}$$

$$\chi_\varepsilon \to \chi \quad \text{weakly star in } H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{3.28}$$

$$\text{and strongly in } C^0\big([0,T];H\big) \cap L^2(0,T;V), \tag{3.29}$$

$$\xi_\varepsilon \to \phi \quad \text{weakly in } L^2(0,T;H). \tag{3.30}$$

Moreover, by (2.2) and (3.29), we have that

$$\sigma'(\chi_\varepsilon) \to \sigma'(\chi) \quad \text{strongly in } C^0\big([0,T];H\big) \tag{3.31}$$

as $\varepsilon \searrow 0$. Moreover, using the fact that $\vartheta_\varepsilon \in \rho(u_\varepsilon)$, where ρ is the maximal monotone graph defined in $(-\infty,0)$ by $\rho(r) = -1/r$ for $r>0$ (cf. also Remark 2.1), (3.25) and (3.27), we have that $\vartheta \in \rho(u)$, simply making use of [2, Proposition 1.1, p. 42]. Analogously, using (3.29), (3.30), and the same proposition, we may recover that $\phi = \xi$. Since it is a standard matter to pass to the limit in $(\mathrm{P}_\varepsilon')$ as $\varepsilon \searrow 0$ and find a solution to Problem (P_0'), which is unique in the sense of Lemma 3.1. With this the proof of Theorem 2.6 is complete.

4. Analysis of Problem (P_ε) in the case $\varepsilon > 0$

We have to analyze now Problem (P_ε) for $\varepsilon > 0$. We follow here the same procedure (and notations) of [13, Section 4] adapting or modifying the results to our case, which needs a more careful analysis because of lack of regularity on χ (due to the fourth order equation (1.2), instead of the second order one of [13]). So, let us first give a uniqueness result on the solution to this problem, which holds also in case $\varepsilon = 0$.

Lemma 4.1. *The components ϑ and χ of solutions to Problem* (P_ε) *are unique for any* $0 \leqslant \varepsilon < 1$.

Proof. Let $0 \leqslant \varepsilon < 1$ be fixed, and (ϑ_i, χ_i), (w_i, ξ_i) $(i = 1, 2)$ two solutions of Problem (P_ε). Put $F_i := f - J(k * \vartheta_i)$ $(i = 1, 2)$ and $F = F_1 - F_2$. Since $F_i \in L^2(0, T; V')$ (cf. (2.20)), it follows that (ϑ_i, χ_i), (w_i, ξ_i) $(i = 1, 2)$ solve (P'_ε) for the right-hand side F_i, $i = 1, 2$. Hence, using the notations of Lemma 3.1, we have that (see (3.5))

$$\big\|\eta(t)\big\|_*^2 + \varepsilon\|\vartheta\|^2_{L^2(0,t;H)} + C_*\big\|\chi(t)\big\|_*^2 + \|\chi\|^2_{L^2(0,t;V)} \leqslant C\|\chi\|^2_{L^2(0,t;V')} - 2\int_0^t \big(J(k*\vartheta), J^{-1}\eta\big), \quad \forall t \in [0, T]. \tag{4.1}$$

Now, thanks to (2.11), (2.35), (2.36), and Schwartz inequality, we have

$$\begin{aligned} -2\int_0^t \big(J(k*\vartheta), J^{-1}\eta\big) &\leqslant -2\int_0^t (k*\vartheta, \vartheta) + 2\ell\|k*\vartheta\|_{L^2(0,t;H)}\|\chi\|_{L^2(0,t;H)} \\ &\leqslant -2\int_0^t (k*\vartheta, \vartheta) + \big(\big|k(0)\big|^2 + \|k'\|^2_{L^1(0,t)}\big)\|1*\vartheta\|^2_{L^2(0,t;H)} \\ &\quad + \ell^2\delta\|\chi\|^2_{L^2(0,t;V)} + C_\delta\|\chi\|^2_{L^2(0,t;V')}, \quad \forall\delta > 0, \end{aligned} \tag{4.2}$$

for all $t \in [0, T]$, and for some positive constant C_δ. Moreover, an integration by parts combined with the use of (2.33) and (2.34) leads to

$$\begin{aligned} I(t) := -2\int_0^t (k*\vartheta, \vartheta) &= -2k(0)\big|(1*\vartheta)(t)\big|^2 - 2\big((k'*1*\vartheta)(t), 1*\vartheta(t)\big) \\ &\quad + 2\int_0^t \frac{1}{2}k(0)\frac{d}{ds}|1*\vartheta|^2(s)\,ds + 2\int_0^t (k'*\vartheta, 1*\vartheta) \\ &= -k(0)\big|(1*\vartheta)(t)\big|^2 - 2\big((k'*1*\vartheta)(t), 1*\vartheta(t)\big) \\ &\quad + 2\int_0^t \big(k'(0)(1*\vartheta) + k''*1*\vartheta, 1*\vartheta\big). \end{aligned} \tag{4.3}$$

Using now (4.1)–(4.3), we may bring the positive term $k(0)|(1*\vartheta)(t)|^2$ (see (2.4)) to the left side and estimate the remaining term on the right. By (2.35), (2.38), and (2.4), we may obtain for all $t \in [0,T]$,

$$
\begin{aligned}
&2\big|\big((k'*1*\vartheta)(t),(1*\vartheta)(t)\big)\big| \\
&\quad\leqslant 2\|k'*1*\vartheta\|_{C^0([0,t];H)}\big|(1*\vartheta)(t)\big| \\
&\quad\leqslant \frac{2}{k(0)}\|k'\|^2_{L^2(0,t)}\|1*\vartheta\|^2_{L^2(0,t;H)}+\frac{k(0)}{2}\big|(1*\vartheta)(t)\big|^2,
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
&2\left|\int_0^t \big(k'(0)(1*\vartheta)+k''*1*\vartheta,1*\vartheta\big)\right| \\
&\quad\leqslant 2\big(\big|k'(0)\big|+\|k''\|_{L^1(0,t)}\big)\|1*\vartheta\|^2_{L^2(0,t;H)}.
\end{aligned}
\tag{4.5}
$$

Finally, collecting (4.1)–(4.5), and choosing δ sufficiently small, we may immediately obtain

$$
\begin{aligned}
&\big\|\eta(t)\big\|_*^2+\varepsilon\|\vartheta\|^2_{L^2(0,t;H)}+\big|(1*\vartheta)(t)\big|^2+\big\|\chi(t)\big\|_*^2+\|\chi\|^2_{L^2(0,t;V)} \\
&\quad\leqslant C\int_0^t \big(\big\|\chi(s)\big\|_*^2+|1*\vartheta|^2\big),\quad \forall t\in[0,T].
\end{aligned}
\tag{4.6}
$$

At this point it suffices to apply Gronwall's lemma to (4.6) in order to conclude the proof of Lemma 4.1. □

Remark 4.2. Let us observe that, if $0<\varepsilon<1$, the same uniqueness result holds for $k\in L^2(0,T)$ instead of k in hypothesis (2.4). In fact, if $\varepsilon>0$, we can estimate the integral in (4.2), by use of (2.35) and (2.38), in this way

$$
-2\int_0^t (k*\vartheta,\vartheta)\leqslant \frac{2}{\varepsilon}\int_0^t \|k\|^2_{L^2(0,T)}\|\vartheta\|^2_{L^2(0,s;H)}\,ds+\frac{\varepsilon}{2}\|\vartheta\|^2_{L^2(0,t;H)},\quad \forall t\in[0,T].
$$

Moreover, observing that $\|1*\vartheta\|^2_{L^2(0,t;H)}\leqslant T\int_0^t\|\vartheta\|^2_{L^2(0,s;H)}\,ds$, Lemma 4.1 still follows by Gronwall's lemma.

Now, we may state some lemmas that will close the proof of Theorem 2.4, i.e., they will give existence of solutions to Problem (P_ε) via fixed point technique.

Lemma 4.3. *Let $0<\varepsilon<1$ and $k\in L^1(0,T)$. Let A_ε be the operator such that $\Theta\mapsto\vartheta$, with $\Theta\in L^2(0,T;V)$ and ϑ the solution component of Problem $(\mathrm{P}'_\varepsilon)$, where F is replaced by $f-J(k*\Theta)$. Then, there holds*

$$
\begin{aligned}
&\|\vartheta\|^2_{H^1(0,t;V')\cap C^0([0,t];H)}+\varepsilon\|\vartheta\|^2_{L^2(0,t;V)} \\
&\quad\leqslant R_1(\varepsilon)+R_2(\varepsilon)\|k\|^2_{L^1(0,t)}\|\Theta\|^2_{L^2(0,t;V)},\quad \forall t\in[0,T],
\end{aligned}
\tag{4.7}
$$

where

$$R_1(\varepsilon) := \bar{C}\big(1+\varepsilon+2(1+\varepsilon^{-1})\|f\|^2_{L^2(0,T;V')}\big), \qquad R_2(\varepsilon) := 2\bar{C}(1+\varepsilon^{-1}) \tag{4.8}$$

for some positive constant $\bar{C}$ depending only on Ω, N, T, ϑ_0, χ_0, and in particular not on ε and $t \in [0,T]$. Moreover, setting $\vartheta_i = A_\varepsilon(\Theta_i)$, $i = 1,2$, there is another constant C_c such that

$$\begin{aligned}\varepsilon\|\vartheta_1-\vartheta_2\|^2_{L^2(0,t;H)} \\ \leqslant C_c(1+\varepsilon^{-1})\|k\|^2_{L^1(0,t)}\|\Theta_1-\Theta_2\|^2_{L^2(0,t;H)}, \quad \forall t \in [0,T].\end{aligned} \tag{4.9}$$

Proof. Let us say that, in order to obtain (4.7), we have to change the first estimate performed on solutions to Problem (P'_ε). In fact, in order to prove the second part of Theorem 2.6, we have taken $F = f$ with f defined in (2.18) and such that (2.5) holds. In such a way, we have been able to get uniform estimates (independent on ε) on solutions of (P_ε). Here instead we do not need estimates independent on ε, because we are just considering Problem (P'_ε) for $\varepsilon > 0$ and so we may take the datum F with the minimal regularity $L^2(0,T;V')$ and change (3.19) as follows:

$$\int_0^t \langle F,\vartheta\rangle \leqslant \frac{\varepsilon}{2}\|\vartheta\|^2_{L^2(0,t;V)} + \frac{1}{2\varepsilon}\|F\|^2_{L^2(0,t;V')}. \tag{4.10}$$

In this case, (3.20) becomes

$$\begin{aligned}&\big|\vartheta(t)\big|^2 + \big\|\ln(\vartheta)(t)\big\|_{L^1(\Omega)} + \|\vartheta_t\|^2_{L^2(0,t;V')} + \|u\|^2_{L^2(0,t;V)} + \varepsilon\|\vartheta\|^2_{L^2(0,t;V)} \\ &\quad + \|\chi_t\|^2_{L^2(0,t;V')} + \big\|\chi(t)\big\|^2 \\ &\leqslant \bar{C}\big(1+\varepsilon+(1+\varepsilon^{-1})\|F\|^2_{L^2(0,t;V')}\big), \quad \forall t \in [0,T],\end{aligned} \tag{4.11}$$

with $\bar{C}$ the same as in the text of our Lemma 4.3. Now, using (2.35), we immediately find (4.7).

Finally, inequality (4.9) is a consequence of (3.5) with $F = -J(k * (\Theta_1 - \Theta_2)$ and $\eta_0 = \chi_0 = 0$, in fact, we can argue as in (4.2), with $\Theta = \Theta_1 - \Theta_2$,

$$\begin{aligned}2\int_0^t \langle F, J^{-1}\eta\rangle &\leqslant (2\varepsilon^{-1}+1)\|k\|^2_{L^1(0,t)}\|\Theta\|^2_{L^2(0,t;H)} \\ &\quad + \frac{\varepsilon}{2}\|\vartheta\|^2_{L^2(0,t;H)} + \ell^2\|\chi\|^2_{L^2(0,t;H)},\end{aligned} \tag{4.12}$$

moreover, applying (2.36) to (4.12), and Gronwall's lemma to (3.5), we just recover (4.9). □

The next two lemmas are formulated as [13, Lemmas 4.4–5], i.e., in the paper where a proof of existence of solutions for an analogous of our Problem (P_ε), in the nonconserved case (i.e., with a second order equation for the phase variable χ) is given. Let us recall them here and give a detailed proof for reader's convenience.

Lemma 4.4. *Let* $0 < \varepsilon < 1$ *and* $k \in L^1(0,T)$. *Then, there exists some* $T_0 \in [0,T]$ *such that* (P_ε) *has a unique solution on* $[0,T_0]$.

Proof. Choose $T_0 > 0$ small enough so that

$$R_2(\varepsilon)\|k\|^2_{L^1(0,T_0)}\frac{2R_1(\varepsilon)}{\varepsilon} \leqslant R_1(\varepsilon) \tag{4.13}$$

and that

$$C_c(1+\varepsilon^{-1})\|k\|^2_{L^1(0,T_0)} \leqslant \frac{\varepsilon}{2}. \tag{4.14}$$

Then A_ε maps the set

$$Y_0 = \left\{v \in L^2(0,T_0;V) \text{ s.t. } \varepsilon\|v\|^2_{L^2(0,T_0;V)} \leqslant 2R_1(\varepsilon)\right\}$$

into itself because of (4.7) together with (4.13). Moreover, Y_0 is a complete metric space if we endow it with the distance

$$d(v_1,v_2) = \|v_1 - v_2\|_{L^2(0,T_0;H)}, \quad v_1, v_2 \in Y_0,$$

and A_ε is strictly contractive in Y_0 thanks to (4.9) and (4.14). And so, by the contractive mapping principle, A_ε has a unique fixed point in Y_0 and so (P_ε) has a unique solution on $[0,T_0]$. □

Lemma 4.5. *Let* $0 < \varepsilon < 1$ *and* $k \in L^2(0,T)$. *Then, Problem* (P_ε) *has a unique solution on* $[0,T]$.

Proof. Thanks to Lemma 4.4 and a standard argument, it suffices to prove an estimate independent on T_0. Observe that a solution (ϑ,χ), (w,ξ) of (P_ε) solves $(\mathrm{P}'_\varepsilon)$ with $F = f - J(k*\vartheta)$. And so, we may recall inequality (4.11), and the estimate

$$\|F\|^2_{L^2(0,t;V')} \leqslant 2\left(\|f\|^2_{L^2(0,T;V')} + \int_0^t \|k\|^2_{L^2(0,T)}\|\vartheta\|^2_{L^2(0,s;V)}\,ds\right), \quad \forall t \in [0,T],$$

obtained using only (2.35). Then, applying Gronwall's lemma, we may conclude the proof. □

With this the proof of Theorem 2.4 is complete.

5. Analysis of Problem (P_0)

In this section we get a solution (ϑ,χ), (w,ξ) to Problem (P_0), whose components ϑ and χ are also unique (see Lemma 4.1). Moreover we find also a continuous dependence result for the solution to Problem (P_ε) with $0 \leqslant \varepsilon < 1$. Let us begin with uniform estimates (independent of ε) on the solution of Problem (P_ε) with $\varepsilon > 0$. We shall denote even in this section by C any positive constant (possibly not the same even inside the same row) that only depends on Ω, N, T, ϑ_0, χ_0, and also on f, but always not on ε and $t \in [0,T]$. Then, we will argue by compactness and find, at the limit for $\varepsilon \searrow 0$, the solution of Problem (P_0).

Third estimate. Let $(\vartheta_\varepsilon, \chi_\varepsilon)$, $(w_\varepsilon, \xi_\varepsilon)$ be the solution of (P_ε) for $\varepsilon > 0$. We modify the first estimate taking $F = f - J(k * \vartheta_\varepsilon)$. Observe that (3.10) may be exactly rewritten with F in place of f. We will still call it (3.10) in the sequel. Now, we may treat the terms which contains F in (3.10) as follows (remember also (2.17)):

$$\int_0^t \bigl\langle F, u_\varepsilon + \delta\mathcal{N}\bigl(\partial_t\vartheta_\varepsilon - (\partial_t\vartheta_\varepsilon)_\Omega\bigr)\bigr\rangle \leqslant C_\star\delta\zeta \bigl\|\partial_t\vartheta_\varepsilon - (\partial_t\vartheta_\varepsilon)_\Omega\bigr\|^2_{L^2(0,T;V')} + \delta C_\zeta \|F\|^2_{L^2(0,T;V')} \tag{5.1}$$

for all $\zeta > 0$ and for some constant $C_\zeta > 0$ (remember that δ was an arbitrary positive constant). Moreover, observe that

$$\|F\|_{L^2(0,T;V')} \leqslant \|f\|_{L^2(0,T;V')} + \bigl(\bigl|k(0)\bigr| + \|k'\|_{L^1(0,T)}\bigr)\int_0^t \bigl\|(1 * \vartheta_\varepsilon)\bigr\|. \tag{5.2}$$

The other term containing F was

$$\delta\int_0^t\int_\Omega F\vartheta_\varepsilon = \delta\int_0^t \langle f, \vartheta_\varepsilon\rangle - \delta\int_0^t \bigl\langle J(k * \vartheta_\varepsilon), \vartheta_\varepsilon\bigr\rangle. \tag{5.3}$$

Now, in order to estimate these last two integrals, we use (2.5) in the first term of (5.3) and integrate by parts the second one with the help of (2.33) and (2.34), getting

$$\int_0^t \langle f, \vartheta_\varepsilon\rangle \leqslant C\left(1 + \int_0^t |\vartheta_\varepsilon|^2\right), \tag{5.4}$$

$$\begin{aligned} -\int_0^t \bigl\langle J(k * \vartheta_\varepsilon), \vartheta_\varepsilon\bigr\rangle &= -\frac{k(0)}{2}\bigl\|1 * \vartheta_\varepsilon(t)\bigr\|^2 - \bigl(\bigl((k' * 1 * \vartheta_\varepsilon)(t), 1 * \vartheta_\varepsilon(t)\bigr)\bigr) \\ &\quad + \int_0^t \bigl(\bigl(k'(0)(1 * \vartheta_\varepsilon) + k'' * 1 * \vartheta_\varepsilon, 1 * \vartheta_\varepsilon\bigr)\bigr). \end{aligned} \tag{5.5}$$

Moreover, to estimate the terms in (5.5), we may proceed exactly like in (4.4) and (4.5), and for the other terms in (3.10) we proceed exactly like in the first estimate (see (3.11)–(3.15) and (3.18)), in such a way that, in place of (3.20), taking also advantages of the second estimate, we get the bound

$$\begin{aligned} &\|\vartheta_\varepsilon\|^2_{H^1(0,T;V')\cap L^\infty(0,T;H)} + \varepsilon\|\vartheta_\varepsilon\|^2_{L^2(0,T;V)} + \|1 * \vartheta_\varepsilon\|^2_{C^0([0,T];V)} \\ &\quad + \|\chi_\varepsilon\|_{H^1(0,T;V')\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\xi_\varepsilon\|_{L^2(0,T;H)} \leqslant C \end{aligned} \tag{5.6}$$

for all $t \in [0, T]$.

Then, we can pass to the limit here for $\varepsilon \searrow 0$ as we have just done in $(\mathrm{P}'_\varepsilon)$ to get solution for (P'_0), noting moreover that, thanks to (5.6), (2.4), and (2.33), we have the additional convergence

$$k * \vartheta_\varepsilon \to k * \vartheta \quad \text{weakly star in } L^\infty(0,T;V). \tag{5.7}$$

With this the proof of Theorem 2.3 is complete.

Now, let us give our continuous dependence result of solution to (P_ε) (for small $\varepsilon \geqslant 0$) on all the data in (2.4)–(2.7), but not on m_0.

Theorem 5.1. *Let* $0 \leqslant \varepsilon < 1$ *and let* (ϑ_i, χ_i), (w_i, ξ_i) $(i = 1, 2)$ *two solutions of* (P_ε) *correspondent to the data* k_i, f_i, ϑ_{0i}, χ_{0i} $(i = 1, 2)$. *Assume notations* (3.1)–(3.3) *and hypothesis* (3.4). *Let moreover*

$$k = k_1 - k_2, \qquad f = f_1 - f_2. \tag{5.8}$$

Then, it holds that

$$\begin{aligned} &\|\eta\|^2_{L^\infty(0,T;V')} + \|\chi\|^2_{L^\infty(0,T;V')\cap L^2(0,T;V)} + \varepsilon\|\vartheta\|^2_{L^2(0,T;H)} + \|1*\vartheta\|^2_{C^0([0,T];H)} \\ &\quad \leqslant C\big(\|\eta_0\|^2_* + \|\chi_0\|^2_* + \|f\|^2_{L^1(0,T;V')} + \|k\|^2_{W^{1,1}(0,T)}\big) \end{aligned} \tag{5.9}$$

for some positive constant C independent on ε.

Proof. We apply (3.5) with $F = f - J(k * \vartheta_1) - J(k_2 * \vartheta)$. Applying Schwartz inequality and (2.10) in the first integral below, and (2.35) in the second one, we get

$$\int_0^t \langle f, J^{-1}\eta\rangle \leqslant \int_0^t \|f\|_*\|\eta\|_*, \tag{5.10}$$

$$\begin{aligned} &-\int_0^t \big\langle J(k*\vartheta_1) + J(k_2*\vartheta), J^{-1}\eta\big\rangle \\ &\quad \leqslant \big(|k(0)| + \|k'\|_{L^1(0,T)}\big)\|1*\vartheta_1\|_{L^2(0,T;V)}\|\eta\|_{L^2(0,T;V')} \\ &\qquad - \int_0^t \big\langle J(k_2*\vartheta), J^{-1}\eta\big\rangle, \end{aligned} \tag{5.11}$$

and now, we may estimate the integral in (5.11) using (2.11), (2.33)–(2.36), and integrating by parts as in (4.3), i.e., we have

$$\begin{aligned} -\int_0^t \big\langle J(k_2*\vartheta), J^{-1}\eta\big\rangle &\leqslant -\int_0^t (k_2*\vartheta, \vartheta) + \ell\|k_2*\vartheta\|_{L^2(0,T;H)}\|\chi\|_{L^2(0,T;H)} \\ &\leqslant \frac{-k_2(0)}{2}\big|(1*\vartheta)(t)\big|^2 - \int_\Omega (k_2'*1*\vartheta)(t)(1*\vartheta)(t) \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \bigl(k_2'(0)(1*\vartheta)+k_2''*1*\vartheta, 1*\vartheta\bigr)\\
&+\bigl(\bigl|k(0)\bigr|^2+\|k_2'\|^2_{L^1(0,T)}\bigr)\|1*\vartheta\|^2_{L^2(0,T;H)}\\
&+2\ell^2\bigl(\delta\|\chi\|^2_{L^2(0,T;V)}+C_\delta\|\chi\|^2_{L^2(0,T;V')}\bigr)
\end{aligned}
\tag{5.12}
$$

for all $\delta > 0$ and for some positive constant C_δ. Moreover, it remains to estimate the two integrals in (5.12). We proceed as follows:

$$
\begin{aligned}
&\left|\int_\Omega (k_2'*1*\vartheta)(t)(1*\vartheta)(t)\right| \leqslant \|k_2'*1*\vartheta\|_{C^0([0,T];H)}\bigl|(1*\vartheta)(t)\bigr|\\
&\quad\leqslant \frac{1}{2k_2(0)}\|k_2'\|^2_{L^2(0,T)}\|1*\vartheta\|^2_{L^2(0,T;H)}+\frac{k_2(0)}{2}\bigl|(1*\vartheta)(t)\bigr|^2,
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
&\left|\int_0^t \bigl(k_2'(0)(1*\vartheta)+k_2''*1*\vartheta, 1*\vartheta\bigr)\right|\\
&\quad\leqslant \bigl(\bigl|k_2'(0)\bigr|+\|k_2''\|_{L^1(0,T)}\bigr)\|1*\vartheta\|^2_{L^2(0,T;H)}.
\end{aligned}
\tag{5.14}
$$

Finally, on account of (2.4), (3.5), (5.6), (5.10)–(5.14), and choosing δ sufficiently small, we obtain

$$
\begin{aligned}
&\bigl\|\eta(t)\bigr\|_*^2+\varepsilon\|\vartheta\|^2_{L^2(0,T;H)}+\frac{k_2(0)}{2}\bigl|(1*\vartheta)(t)\bigr|^2+C_*\bigl\|\chi(t)\bigr\|_*^2+C\|\chi\|^2_{L^2(0,T;V)}\\
&\quad\leqslant \|\eta_0\|_*^2+C_\star\|\chi_0\|_*^2+\int_0^t \|f\|_*\|\eta\|_*+C\bigl(\bigl|k(0)\bigr|+\|k'\|_{L^1(0,T)}\bigr)^2\\
&\qquad+C\int_0^t\bigl((1+\ell^2)\|\chi\|_*^2+|1*\vartheta|^2\bigr),\quad \forall t\in[0,T].
\end{aligned}
\tag{5.15}
$$

At this point, we may apply Gronwall's lemma and deduce (5.9). $\square$

Remark 5.2. Let us point out that Theorem 5.1 is of some utility since it gives us furthermore the possibility of defining yet a weaker solution of Problem (P_ε), for small $\varepsilon \geqslant 0$ when the data (f,k) are only assumed to be in $L^1(0,T;V')\times W^{1,1}(0,T)$, with $k(0)>0$, and (η_0,χ_0) in $V'\times V'$. Indeed, a consequence of (5.9) is that it would be sufficient to approximate (f,k) and (η_0,χ_0) in the above norms by some families $f_\delta, k_\delta, \eta_{0\delta}, \chi_{0\delta}$ such that

$$
f_\delta\in L^2(0,T;H),\quad k_\delta\in W^{2,1}(0,T),\quad k_\delta(0)>0,\quad \eta_{0\delta}\in H,\quad \chi_{0\delta}\in V
$$

for all $\delta > 0$, then devising a Cauchy argument. In this case we would obtain a solution (ϑ,χ) as limit of sequence of approximating solutions, with the regularities

$$
\vartheta\in L^\infty(0,T;V'),\qquad \chi\in L^\infty(0,T;V')\cap L^2(0,T;V),
$$

and

$$1 * \vartheta \in C^0\big([0,T]; H\big) \quad (\text{with } \vartheta \in L^2(0,T;H) \text{ only if } \varepsilon > 0).$$

Moreover, this solution is unique in the class of such limiting solutions and depends continuously on the data.

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